North Pole Adventures

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Abstract

This paper describes how historical events more than 70 years ago still provide enough inspiration to build challenging problems. These problems uncover various possible areas for the application of Mathematics, Computer Science and Digital Art. The paper is focused on presentation of a variety of curves generated by imaginary flights to and from the North Pole and some of the key factors, which determine the shape of these curves. The author also presents a Logo-based microworld – The North Pole – that is used to explore numerous possibilities and that can be modified by used in order to study more complex flights and conditions.

Keywords

Elica, Exploration model, Logo-based microworld, North Pole

Introduction

Most of the time people successfully solve problems but there are cases when the goal is just the opposite - to make rather than solve problems. Although it sounds bizarre, the ability to make good problems is one of the self-stimuli of humans.

This paper describes how history can provide us with ideas for creating challenging problems. Although at the beginning these ideas are simple, they later evolve into the basis for a broad field of research, inspired partially by the possibilities for exploration afforded by a Logo-based microworld environment. Such microworlds might also encourage students to make and explore mathematical problems, as well as problems in the areas of computer science or even digital art. That's why the paper does not go beyond describing some interesting observations and does not provide solutions to the problems. It is relatively light on formal and algebraic representations and it simply presents a variety of curves generated via procedural experiments, and suggests conjectures to explore further.

So, let's start the adventure.

On May 9, 1926, two persons, the navigator lieutenant commander Richard Byrd and the pilot Floyd Bennett announced that they were the first to fly an airplane over the North Pole (Goerler and Cullather 1996). The journey started shortly after midnight from King's Bay, Spitsbergen, Norway, and few hours later they passed over the geographical North Pole. Both men were awarded the Medal of Honor after their

"At 9:02 a.m., May 9, 1926, Greenwich civil time, our calculations showed us to be at the Pole! The dream of our lifetime had at last been realized. We headed to the right to take two confirming sights of the sun, then turned and took two more. After that we made some moving and still pictures, then went on for several miles in the direction we had come, and made another larger circle to be sure to take in the Pole. We thus made a non-stop flight around the world in a very few minutes."
From Skyward,
by Rear Admiral Richard E. Byrd, 1928

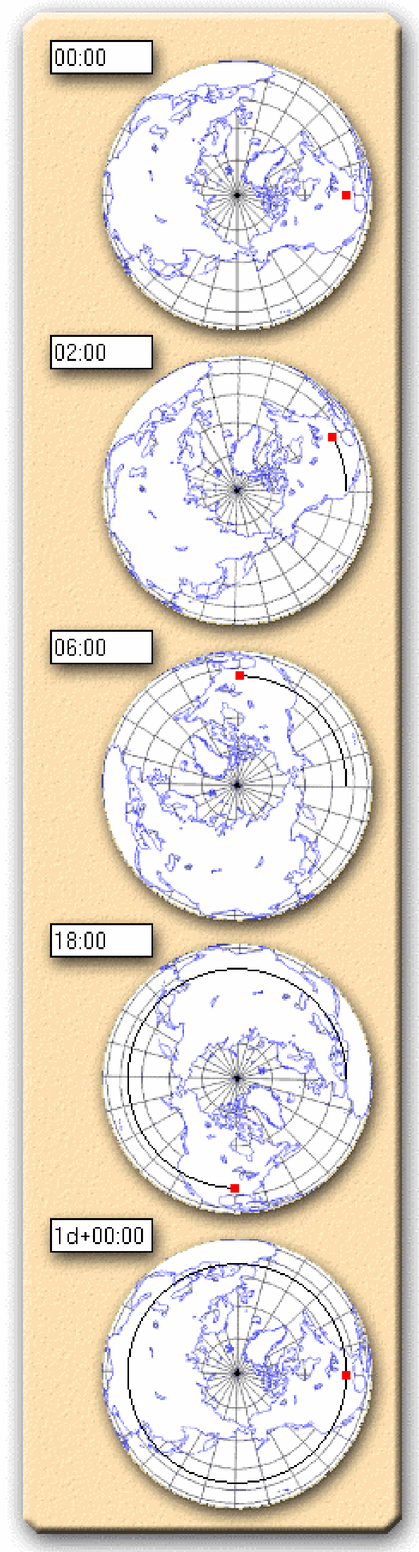
return to the United States. Three years later, in 1929, Byrd crossed the Southern Pole also. Apparently, some researchers (Molett) say their record is questionable.

Byrd's flight took around 16 hours - eight hours to the Pole and eight to return. Imagine we want to monitor such a flight. Where should we put the monitoring station or viewpoint in order to best view the trajectory of the flight? We could easily imagine the path of the flight from different possible locations. Some viewpoints, namely the take-off point and cockpit are not interesting, because the flight would look like a segment due to the fact that the Earth rotation is not considered. This leads us to the assumption that we must consider the rotation. Thus, a place somewhere above the North Pole (About 2002) sounds reasonably interesting, as long as our intuition or common sense could hardly answer the following question:

What path will the airplane follow?

This very question is the one that gave birth to this paper. Of course, there are others, which sprang up later. Does the speed of the airplane matter? How and why? Does the place of take off matter as well? What will be the effect if the flight does not stop at the Pole but continues forward?

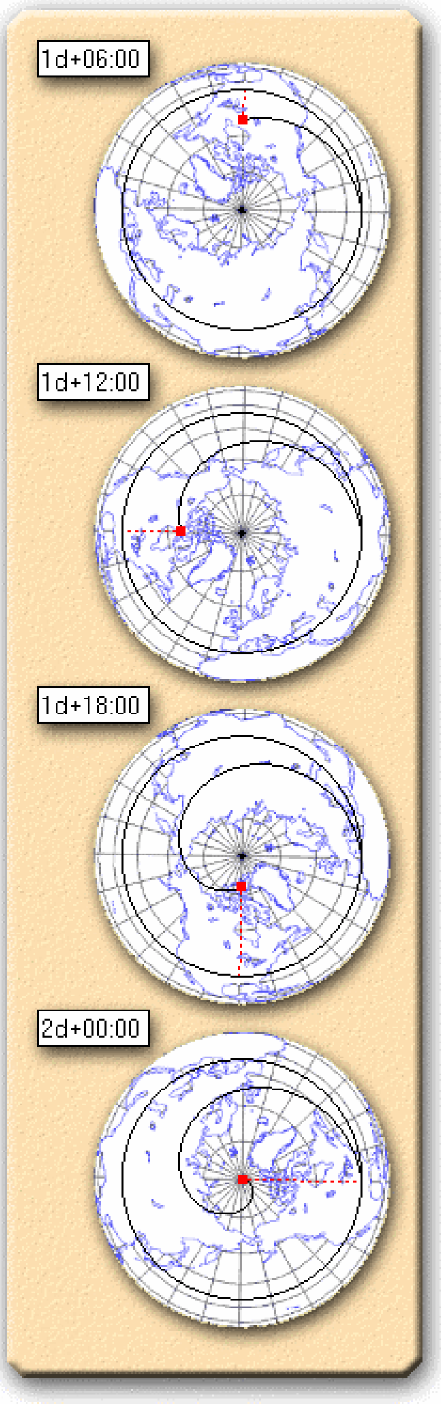
To facilitate the experiments, we have created a new Logo-based microworld. It is called ***The North Pole Project*** and is built using ***Elica*** – the Educational Logo Interface for Creative Activities (Boytchev 2001a). We will visualize the airplane as a small square dot (it is red, if the print is in color) and its path will be a thin black curve. We would also like to see the relative movement of the airplane with respect to the surface. It will be rendered as a dotted segment starting from the take-off point and ending at the current airplane position. And finally, our take-off point will be in North America, rather than in Europe.



Picture 1. Airplane does not fly

Let's start from the simplest case i.e. when the airplane does not fly. The series of snapshots rendered by Elica show that the dot will turn around the Pole, because of the Earth rotation – picture 1. We can easily figure out the elapsed time if we cut the Earth's disk on 24 equal slices - each of them will represent one hour.

The images from the series represent the situation after 2, 6, 18 and 24 hours. Because of the Earth rotation the black arc tracing the airplane path becomes a circle.



Picture 2. A flight to the pole for 24 hours

Let's make a more realistic experiment. Let the airplane fly with a constant speed to the North Pole and reach it in 24 hours. For the first 6 hours its path will look like an arc, but the dot will be closer to the Pole – picture 2.

A reasonable expectation might be that the new path makes a smaller circle. To check it we continue the experiment for another 6 hours. The second set of images shows that the new path is definitely not a circle.

To find out whether it is a part of any conical curve (like ellipses, parabolas or hyperbolas) or it is another type of curve, we continue to investigate. After 6 hours the curve will start to resemble a spiral and in 6 more hours it will strongly resemble a spiral. Airplane speed is calculated to reach the destination in exactly 24 hours. Therefore, we are sure that at the end the airplane will be on the Pole.

What we have experimented with so far can be easily expressed in polar coordinates (Claeys 1997; Weisstein 1999a; Kleitman 2001). The path forms a spiral commonly known as Archimedes' spiral (Gray 1997 pp. 90-92; Lockwood 1967 p. 175; Pappas 1989 p. 149). The polar equation of the airplane path is:

(1) $\boldsymbol{r = r_0 - a\varphi}$,

where $\boldsymbol{r}$ stands for the distance from the airplane to the North Pole, $\boldsymbol{r_0}$ is the distance from the take-off to the North Pole, $\boldsymbol{\varphi}$ is the polar angle representing the rotation speed and thus giving the elapsed time (2π=24h); and $\boldsymbol{a}$ is a constant representing the airplane speed with respect to Earth surface. At the time of landing $\boldsymbol{r = 0}$, and $\boldsymbol{\varphi = 2\pi}$, thus we get:

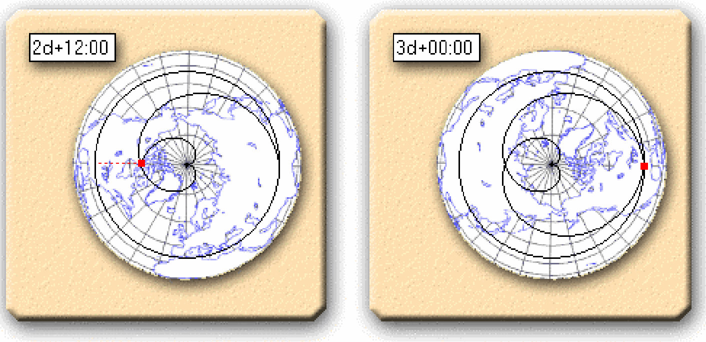
(2) $\boldsymbol{a = r_0/2\pi}$, and

(3) $\boldsymbol{r = r_0 - r_0\varphi / 2\pi}$.

Let's now fly back to the take-off place. Returning takes another 24 hours. Going home will bring us the first surprise. Predicting the path of the plane right now will be a hard test of our intuition.

We can expect that the path will continue to be a spiral. That's natural and can be explained. The polar equation for the returning path is:

(4) $r = a\varphi = r_0\varphi / 2\pi$,

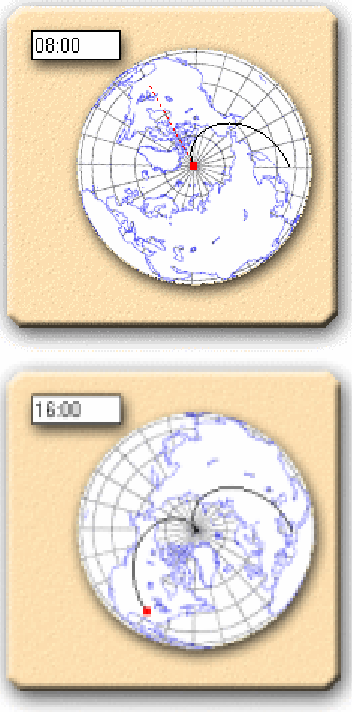


Picture 3. Flying back for 24 hours

where a is the same as in (2).

Exactly 12 hours after we start to fly back, our path will cross itself for the first time. Except for the lovely heart, what we get here is the deep feeling…of symmetry. The paths during the12 hours just before reaching the Pole and the 12 hours just after taking off from the Pole look the same. Why both paths are symmetrical is a good topic for further reflection.

Let's go on. The next 12 hours are full of expectations - will the picture become entirely symmetrical? The answer is on the second image of picture 3.



Picture 4. Flying at the same speed as in Byrd's flight

In respect to history we might want to see the actual path at Byrd's speed. He flew 3 times faster and reached the pole for 8 hours only.

(5) $r = r_0 - a_{Byrd}2\pi/3 = 0$

(6) $a_{Byrd} = 3r_0 / 2\pi = 3a$

The latter gives us a direct linear representation of airplane speed as a function of the initial a, which is $r_0/2\pi$.

The images in picture 4 show the path at reaching the Pole and after the return when the speed is the same as Byrd's. The second half is symmetrical to the first one and it does not close the trajectory.

As we see from the examples, the visual representation of the path depends on the speed, while its representation as polar equations gives no direct hints about this dependence.

Speed does matter

All the considerations so far could be done by performing mental experiments. In order to play with various speeds it is very helpful to use the Elica North Pole microworld. It gives us the flexibility to easily carry out many experiments and it helps us determine what cases are fruitful for further research.

Fortunately, there appear to be many such cases.

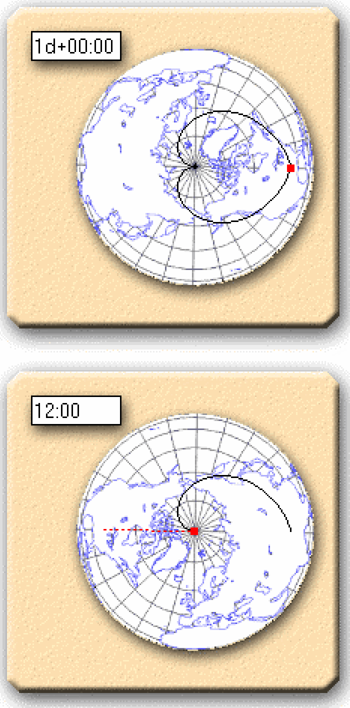
Let's first explore the situation of going the Pole and returning back in a day. Earth orientation will be the same after 24 hours. The position of the airplane in relation to the Earth will be the same too. That is true because we want the airplane to return to its initial position in exactly 24 hours. We can also expect that the path will be closed.

The first 12 hours will be spent on flying to the Pole. The next 12 will be spent on flying back. We call each of these flights ***a trip***. Thus, going to the Pole is one trip, going back - another one. The trip is also used as a distance unit equal to r_0. This helps us define a new unit of speed - ***hours per trip***. The speed of an airplane that reaches the North Pole for 12 hours is ***12 hpt***. We will call this speed ***a base speed*** a_1, and it can be formalized by

(7) $a_1 = 2a = r_0/\pi$

We can represent other speeds as functions of a_1. A speed which is n times faster than the base one, is:

(8) $a_n = na_1 = nr_0/\pi = 12/n$ ***hpt***



Picture 5. Flight to the North Pole and back at base speed a_1

Let's go back to the experiment. Picture 5 shows the path at the base speed after the first and the second trips.

Fortunately or not, it is *not a* cardioid (Beyer 1987 p. 214; Gray 1997 pp. 54-55; Lockwood 1967 pp. 34-43; Yates 1952 pp.4-7) with a polar equation:

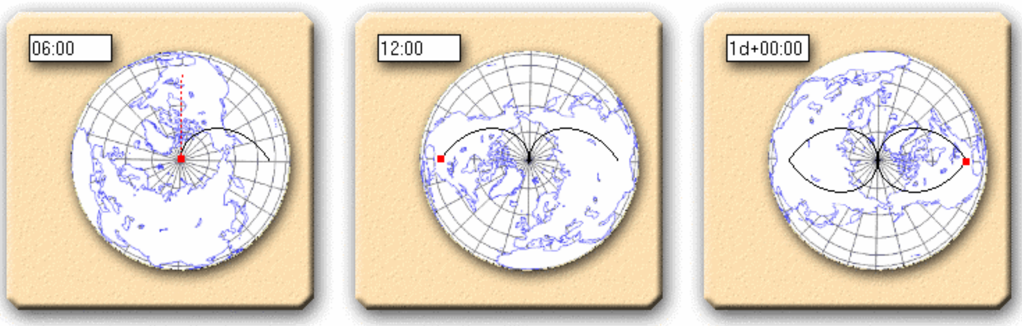
(9) $r = a(1+ \cos\varphi)$

at least for two reasons:

- The Archimedes' spiral is a linear function of the polar angle, while the cardioid is not, and
- The airplane path is not a smooth curve at the point of closure, while the cardioid is smooth at that part of the curve

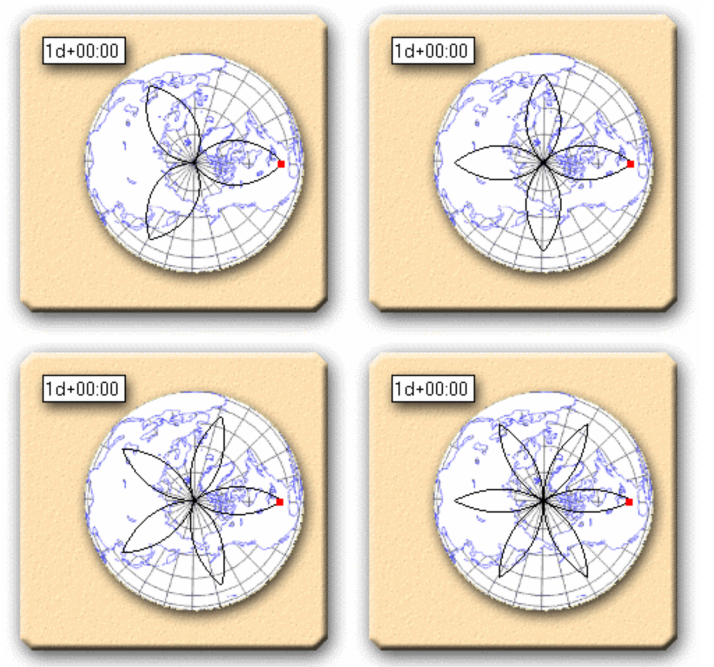
Although many of the paths shown in this paper appear to be quite similar to some curves studied with polar graphs (Maple 2001), they are not.

Let's make the speed twice as fast – $2a_1$ (= 6 hpt). The path after the first trip will not be closed. It will not be closed even after the second trip. To close it we have to go two more trips – picture 6. For further experiments, we assume that the airplane makes as many trips as it is necessary to close the path.



Picture 6. Flight at speed a_2

Let's continue exploring with other speeds. Three times higher speed $3a_1$ (= 4 hpt) results in a 3-leaf path. If the speed is $4a_1$ (= 3 hpt) the leaves are 4 – picture 7. If the airplane flies at $5a_1$ (= 2.4 hpt) we are supposed to see 5 leaves, and at $6a_1$ (= 2 hpt) we expect a 6-leaf path.



Picture 7. Flights at speeds a_3, a_4, a_5 and a_6.

To have a closed path, the number of trips must be even. If the airplane flies at speed a_n, and its speed is 12/n hpt, then for 24 hours (one turn of the Earth) it will make 2n trips which will form exactly n leaves.

Investigations lead to interesting cases, especially when the speed is greater by a fractional number of times. These cases will clearly demonstrate that the above conclusion is generally true for positive integer numbers only.

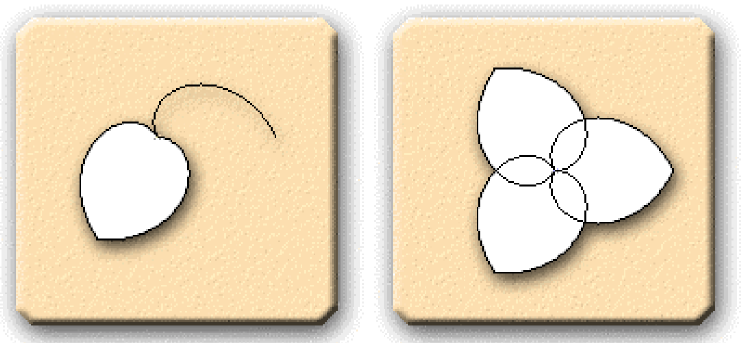
Fractional Speeds

Our further explorations will deal with fractional numbers. For $a_{1.5}$ we get:

(10) $a_{1.5} = 12/1.5 \text{ hpt} = 8 \text{hpt} = a_{Byrd}$,

which is the speed of Byrd's airplane. Although we already know the path, we will continue it until it is closed. For 24 hours, we can make 3 trips and there will be a leaf and a half in the path.

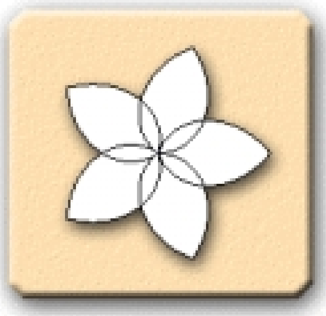
The first image of picture 8 shows how the path looks like after the first day - a leaf and a half (for clarity the image of the Earth is removed as well as the clock at the upper right corner).



Picture 8. Flight at a fractional speed $a_{1.5}$

The leaf is composed of three arcs - one for the handle (this is the first trip) and two for the leaf (the second and the third trips). If we make 3 more trips, we will have 3 leaves and a closed path.

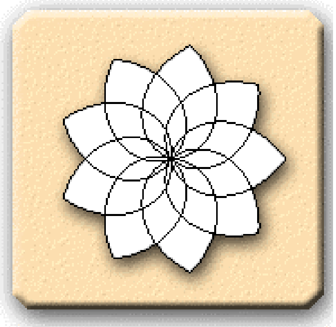
Let's try now with another fractional speed: $a_{2.5}$ (= 4.8 hpt). The result is a flower with 5 overlapping leaves – picture 9.



Picture 9. Flight at speed $a_{2.5}$

If we study speeds with n=k$^1/_2$ (***a*** $_{1.5}$, ***a*** $_{2.5}$, ***a*** $_{3.5}$ etc.) we would see that the number of leaves is always 2n = 2k+1.

Consider a 2$^1/_4$ times greater speed (***a*** $_{2.25}$). The corresponding image (picture 10) has 9 leaves. After a similar investigation on n=k$^1/_4$ (***a*** $_{1.25}$, ***a*** $_{2.25}$, ***a*** $_{3.25}$ etc.) we would see that the number of leaves is always 4n = 4k+1. This leads to the generalization that the number of leaves is the product of the speed factor by the smallest positive integer, so that the product is an integer too. Whether it is true for any rational speed, is a nice challenge for further mathematical explorations, that go beyond the scope of this paper.

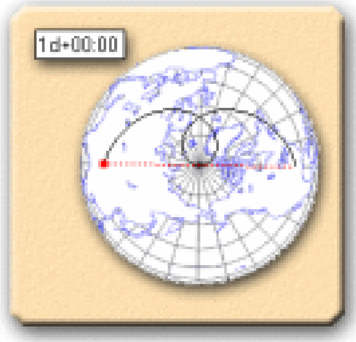


Picture 10. Flight at speed

Going Forward

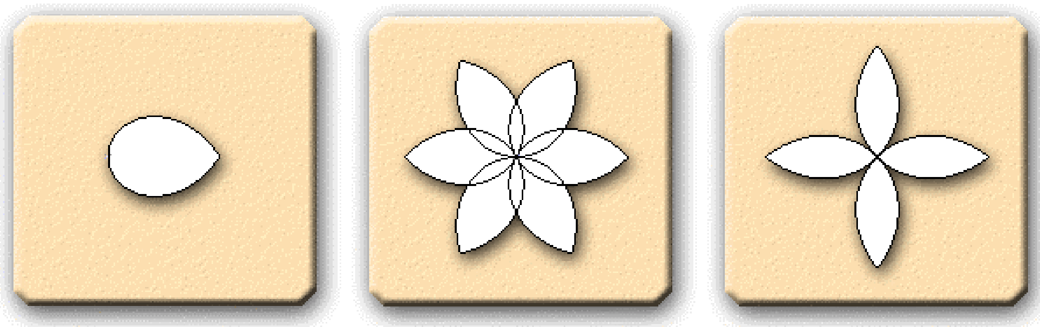
Imagine the airplane flies to the North Pole and continues to fly forward instead of going back. Too see whether the path looks different let's examine what happens at the basic speed ***a*** $_1$. The pilot needs 12 hours for the first trip to get to the North Pole and another 12 hours to go to the *opposite* point of our starting position (a place somewhere in Asia).

The flight has the same time duration as the heart-like one, except that it does not turn back over the North Pole. As seen from picture 11, going forward significantly changes the path.



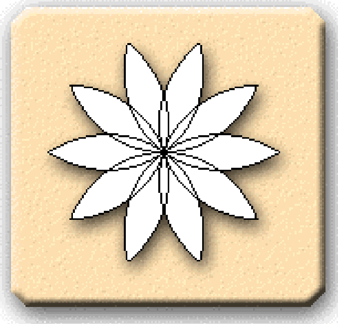
Picture 11. Flight over the North Pole up to somewhere in Asia at speed a_1

Let's explore what happens at ***a*** $_2$ (= 6 hpt) – picture 12. Surprisingly, the image looks like a single leaf. If the airplane flies at ***a*** $_3$ (= 4 hpt), which is three times faster, then the path will form six overlapping leaves! The shock continues at ***a*** $_4$ (= 3 hpt) where the picture is the same as if we fly only to the North Pole and backwards. We get 4 leaves only, but they are not overlapping.



Picture 12. Flight to Asia at speeds a_2, a_3 and a_4.

Just to encourage further explorations, we present the image of the path at five times higher speed a_5 (= $2^2/_5$ hpt) – picture 13.

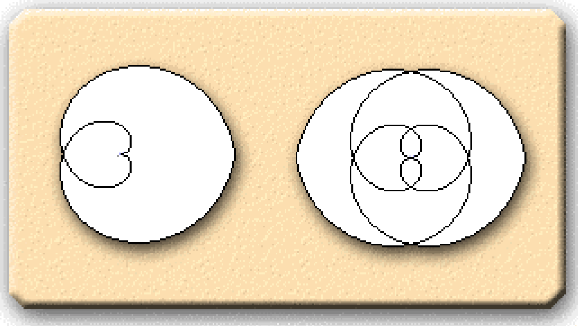


Picture 13. Flight to Asia at a_5

Slow Down

Another possible direction for exploration is studying cases at lower speeds. They generate a greater variety of paths.

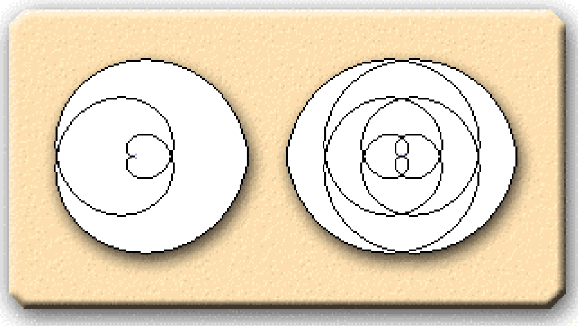
All the images in this section are in pairs, corresponding to two kinds of flights. The left one is for flights to the North Pole and back. The right one is for flights to Asia via the Pole, and back.



Picture 14. Flight to the North Pole and to Asia at $a_{1/2}$

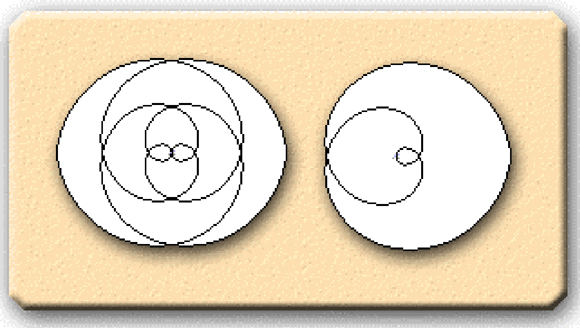
Let's start with a half the speed of the base one: $a_{1/2}$ (= 24 hpt). The images look quite different. The first one has a heart and the right one is like a knit-puzzle – picture 14.

We may go on with three times lower speed $a_{1/3}$ (= 36 hpt) and notice that both pairs of images look alike. We may even think that there is some relationship between different speeds – picture 15.

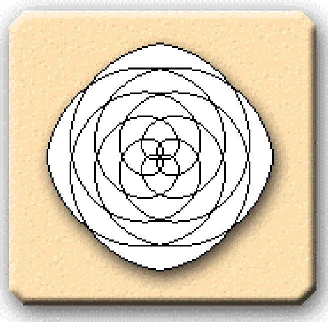


Picture 15. Flight to the North Pole and to Asia at $a_{1/3}$

Continuing with the examples, we will see one intermediate possition in picture 16.It is for a flight at $a_{1/2.5}$. and it appears as if both paths have exchanged their shapes! Our last example is another interesting case. At speed $a_{1/2.25}$ it turns out that both images are identical – picture 17.

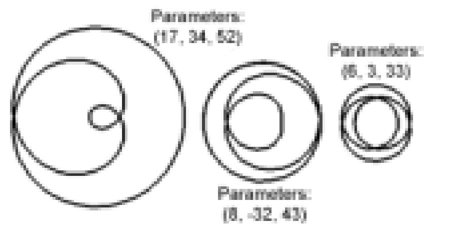


Picture 16. Flight to the North Pole and to Asia at $a_{1/2.5}$



Picture 17. Flight to Asia at

When generating some of the paths, it has been found that some of them look like the designs made up with spirographs (Garg 2002; Levy 1995; Weisstein 1999b) shown in picture 18.



Picture 18. Spiralograph curves

Spirograph curves, however, although similar, are not the same as the paths of the airplane.

Further exploration could include testing and experimenting with various parameters of the flight – picture 19.

The collection of paths is arranged in 4 columns. The first one is for flying to the North Pole with speed greater than the base one by factors of 1, $1^{1}/_{4}$, $1^{1}/_{2}$, $1^{3}/_{4}$, … 4.

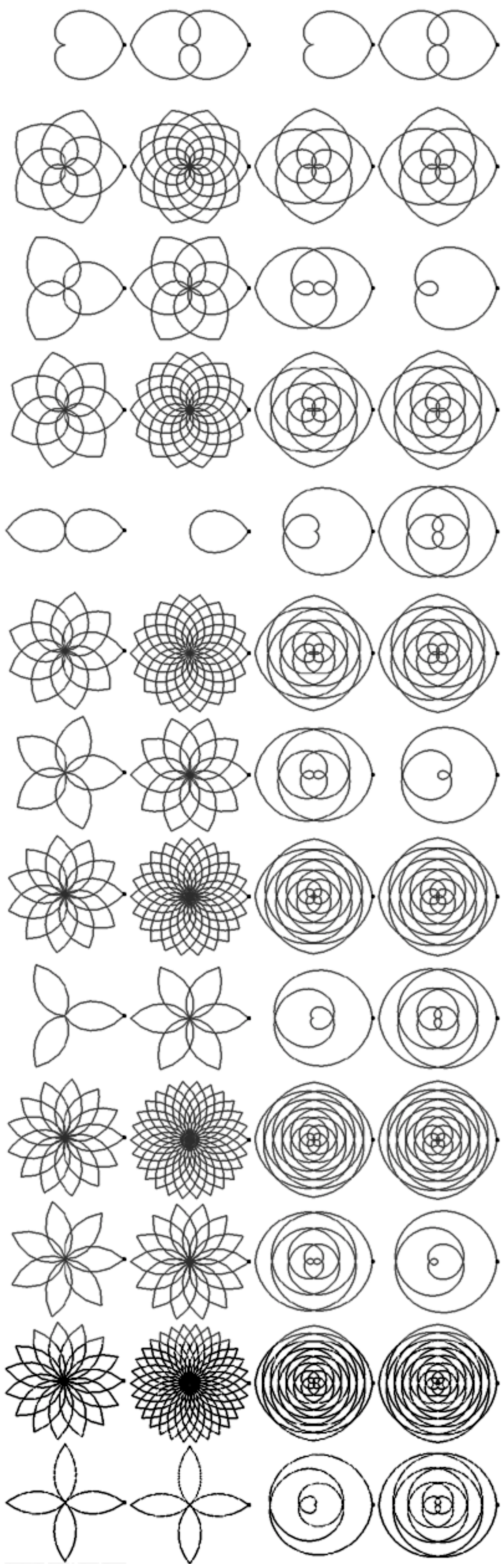
The second column contains another 13 paths, which correspond to flight to Asia. Speed factors are the same as in the first column.

The last two columns are similar to the first two except that they represent lower speeds by the same factors.

The Art of Flying (or The Flying Art)

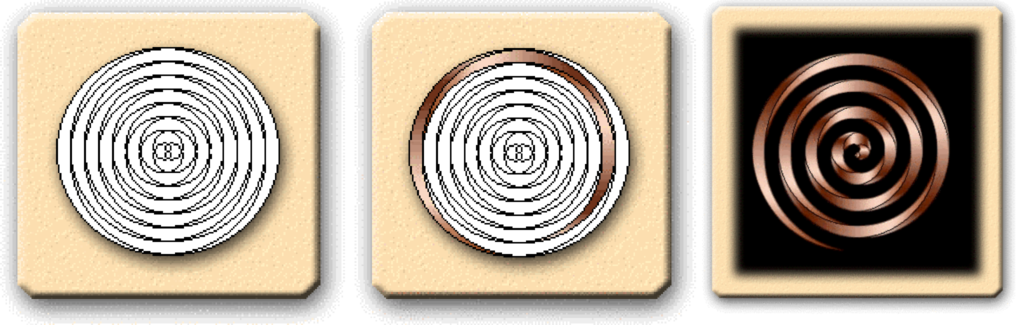
Historically, the first application of the beautiful paths was in the context of some art activities. A path can be treated as a grid or a frame that cuts the plane into segments. A proper coloring and shading of these segments may turn them into pieces of a really nice picture.

Although the Elica North Pole microworld does not render any coloring, it is easy to copy generated grids into a specialized image processing software or make printouts and color them manually.



Picture 19. A comparison table of flight paths at various speed factors starting from 1 to 4 (inclusively) by steps of $^{1}/_{4}$.

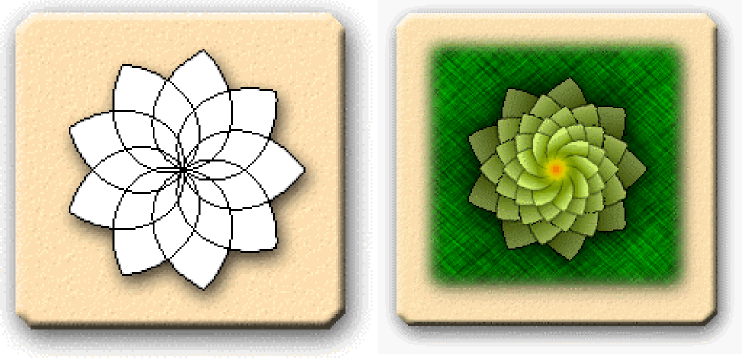
Let’s consider the following scenario. The airplane goes to the opposite point, comes back and then repeats this until the path is closed. The speed is set to 96 hpt, which generates the leftmost image in picture 20.



Picture 20. Phases of coloring a 96 hpt path

The next phase of our artistic activity is to select some of the sectors and gradient-fill them in an appropriate way. We start from the left-most one, move clockwise and at the end of a sector we continue with the inner adjacent sector.

When we color all selected sectors and wipe the rest of the image to finally get an impressive effect.

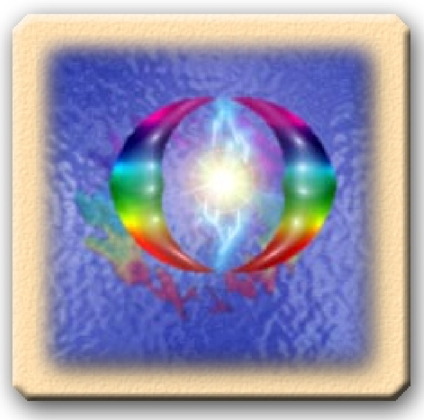


Picture 21. Tropical Flower

The second example of path coloring is the making of a tropical flower. The left image of picture 21 is the path at speed $5^{1}/_{3}$ hpt. If we extract the path, clone it a couple of times, then scale clones, and finally color them, we could "produce" an image of that plant.

Let's look at a more futuristic example (picture 22), which is based on a specific path obtained when we explored the case of lower speeds.

Only two of the sectors of the complete path are used here. They determine the shape of a UFO-like vehicle. The underlying texture of ocean waves and the rippled reflection of the spaceship gives the feeling of hovering.



Picture 22. UFO

Filling grids in different colors is not the limit of their artistic use. The ceiling is set by our imagination only.

The last example – picture 23 – is a stone plate with strange unknown script. Most of the letters in the text are from unknown alphabet. The trick is that each letter from the script comes from the path of a specific flight to, around or via the North Pole.

Some letters require more complex movements like a combination of shorter flights and waiting periods. Other letters require more sophisticated tricks.

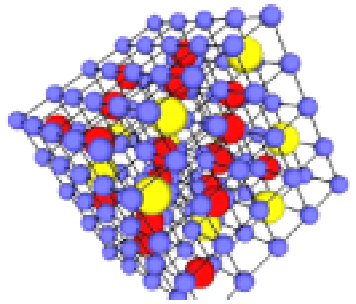
The script, shown here, is just an illustration. It is not reconstruction of a known alphabet or a script, and is, therefore, linguistically meaningless.

Picture 23. The Secret Script

About Elica and The Elica North Pole Microworld

Investigating all various flight scenarios is not a hard work if we have a specially tuned microworld. Logo is perfect for building such environments, which provide a rich source for playing with ideas in mathematics, art and computer science. For the North Pole Microworld we have used Elica Logo (Boytchev 1999).

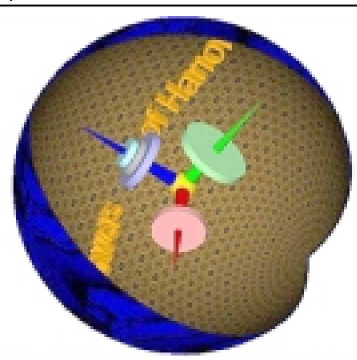
The name of Elica is an acronym for Educational Logo Interface for Creative Activities. Initially Elica's predecessors have been designed to enrich another Logo implementation, namely Geomland – the former Plane Geometry System (Sendov 1998). Later, Elica grew to the level that Geomland has been implemented as a library. Even more, the Logo language itself has been made a library too.



Picture 24. Imaginary Crystal

Nowadays, the core of Elica supports less than a dozen of reserved words (Boytchev 2000). Although physically limited, the core makes it easy to build various microworlds like Logo and Geomland. Thus Elica can be treated as a fundamental microworld for building other microworlds, like Logo, that are used to build more sophisticated microworlds like Geomland and The North Pole.

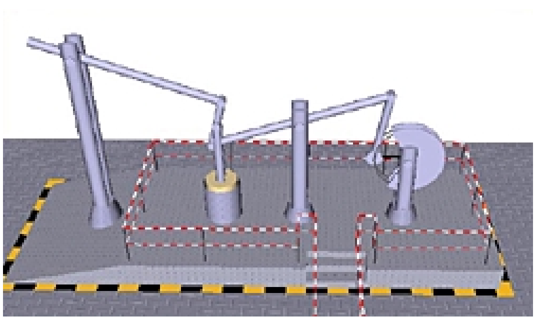
Elica core supports object oriented programming, most aspects of functional programming, and some features of logical programming. There are various other microworlds built on and in Elica. Some of them explore 3D modelling and animation, others are focused on simulations. Elica applications are used in developing courseware, games, as well as in all activities unique to Logo languages. Pictures 24, 25, 26 and 27 are all snapshots from some Elica applications. The imaginary crystal in pricture 24 shows that Elica can be used to build 3D model of crystals and molecules. The next pixture demonstrates a 3D orthogonal version of the famous puzzle "Tower of Hanoi" (Spencer 1998, Bogomolny 1996) widely used in Computer Science introductory courses.



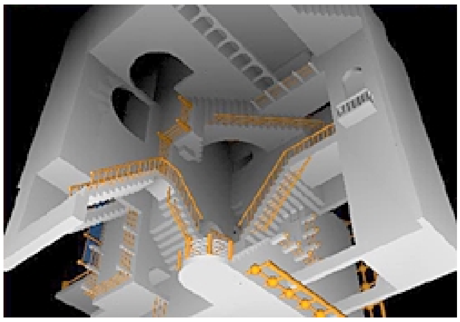
Picture 25. Tower of Hanoi in 3D

Picture 26 shows a snapshot of a steam machine model. The initial 2D model is ported from Geomland. The Elica modification of the same program animates all moving part of the model, while at the same time its elements are converted from flat wireframe objects into a 3D solids. The last example, shown on picture 27, is a reconstruction of a virtual illusionary house inspired by works of Escher (Escher 1995).

You can view a larger set of snapshots in The Elica Museum page (Boytchev 2001b)



Picture 26. Steam machine model



Picture 27. A-la-Escher

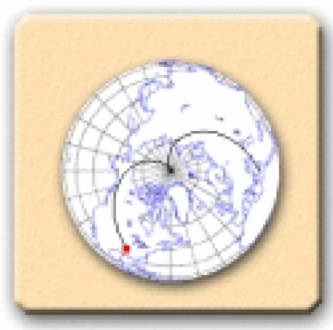
or learn more about the applications in The Elica Application page (Boytchev 2001c).

The Elica North Pole microworld is built to simplify the exploration process by means of suitable language extension and a graphical visualization. It has been decided that the language extension introduced by the microworld must integrate with the existing commands and to be somewhat closer to the natural language.

The following table contains the commands introduced by the North Pole microworld.

North Pole microworld Commands

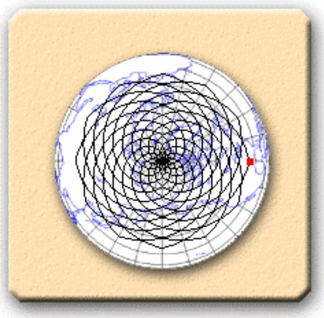
Command	Meaning:
n hpt	Sets the speed to ***n*** hpt. Hpt is as suffix operator.
start from ***t***	Sets airplane position to ***t***, where ***t*** is ***A*** for USA, ***N*** for the North Pole, ***B*** for Asia, or any intermediate number from -1 to 1 where 1 represents point A, 0 represents N and -1 represents B. Thus 0.5 represents halfway between A and N.
land in ***t***	Flies the airplane to position ***t***.
wait ***h*** hour wait ***h*** hours wait for ***h*** hour wait for ***h*** hours	Stops the airplane for ***h*** hours, while the Earth continues to rotate.
remove	Removes the image of the Earth for faster rendering of paths.
bringback	Brings back the image of the Earthfor better visualization.
restart	Initializes airplane position and clears all rendered paths.



Picture 28. Example at 8 hpt

Not only does the microworld accept these commands but they are treated as real Logo commands, i.e. they can be embedded among other ordinary Logo commands. Without this Elica feature, one is expected to write one's own mini-interpreter.

The functionality of the microworld's command is best demonstrated by examples. In the next examples all commands that are introduced by the North Pole Project are in bold. The first example (picture 28) is of a flight to the Pole and back at Byrd's speed of 8 hpt.

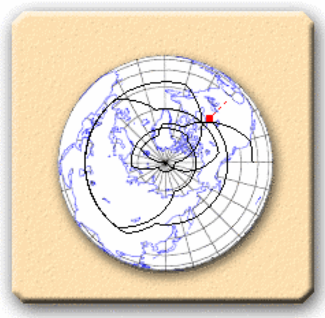

Picture 29. Logo and North Pole microworld

```
8 hpt
land in N
land in A
```

A more complex example is to combine North Pole's commands among standard Logo commands. Picture 29 can be reproduced by the next program, which simulates 24 trips at 17 hpt:

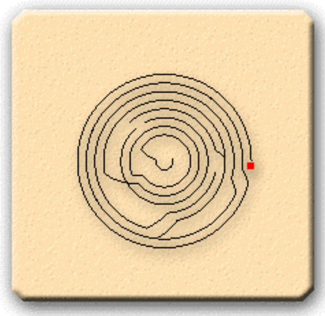
```
17 hpt
repeat 12
   [ land in N
     land in A
   ]
```

Mixing new and Logo commands can be extended to a level at which the user writes a full-features Logo programs to control the flight. In the next example, the command "random" is used to generate a random number from 0 to 1. This command is defined by Elica Logo library. The example itself generates a complex flight with 20 intermediate stops and random places of landing.


Picture 30. Random flights

```
make "i 0
repeat 20
  [ make "i :i+1
    wait :i hours
    land in random

  ]
```


Picture 31. Flight labyrinth

The last example traces an airplane path controlled by lists of fligh data. Each list contains one or more pairs of numbers. The first number if the amout of time to wait. The other is the place to fly to. The execution of the example generate a small labyrinth shown in picture 31.

```
to wait_and_land :data
   start from first :data
   make "data bf :data
   while :data<>[]
   [
     if :data<>[] [wait for (first :data) hours]
     make "data bf :data
     if :data<>[] [land in first :data]
     make "data bf :data
   ]
end
remove
wait_and_land [1 23 0.9 18 0.7 18 0.4 23 0.3 20 0.1 14]
wait_and_land [0.9 21 0.8 18 0.6 5]
wait_and_land [0.7 1 0.6 8]
wait_and_land [0.5 23]
wait_and_land [0.9 12]
wait_and_land [0.7 4]
start from A
```

Other explorations

It is not possible to cover even briefly all possible further explorations based on the North Pole microworld. It is possible to get even more interesting results, if we go beyond the limitation of this specific microworld. Here are just few of them:

a) The airplane might stop at some places and wait for a predefined number of hours. Although this option is not covered in the paper, it is supported in the microworld.

b) The speed of the airplane (and/or Earth rotation) might be a function of time. This will make it possible to generate other spirals.

c) The airplane might not fly along a straight line, but along a curve. Thus some paths might become hypocycloids (Yates 1952) or other curves.

d) The path might be studied not as a planar curve, but in 3D. The airplane might fly between South Pole and North Pole forming curves like spherical spirals (Lauwerier 1991, pp. 64-66), or the flight might be observed from the Moon.

e) Another “central” point might be used instead of the North Pole. It might be either fixed or move as a function of time

f) The Earth might be replaced by a nonspherical surface.

Of course, it is not compulsory to explore only modifications of the initial problem environment. One could focus on detailed mathematical analysis about the trajectory and different factors that influence it.

Others could like to pay attention mainly to developing a more complex and powerful microworld. A key factor in building microworlds is to allow their users to explore various ideas from theoretical considerations to experiments.

The microworld presented in this paper is a model, which shows how to explore new directions. By extending the limits new paths for exploration emerge. The model both uncovers needed explanations and facilitates discovery of new domains.

When supported by appropriate curriculum activities, educators may use the North Pole microworld to illustrate that the purpose of mathematics and computer science is not only to solve problems, but also to formulate new ones.

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